Lecture III

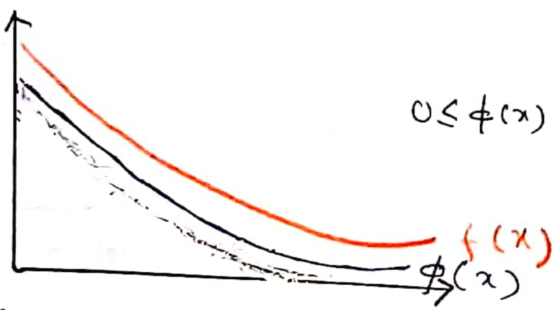
Test for the Convergence of $\int_c^\infty f(x)\,dx$.

(A) Comparison Test

Let $f(x)$ and $\phi(x)$ be two functions which are bounded and integrable in the interval (c, ∞). Also let $f(x)$ be positive and $|\phi(x)| \leq f(x)$ when $x \geq c$. Then if $\int_c^\infty f(x)\,dx$ is Convergent, it follows that $\int_c^\infty \phi(x)\,dx$ is also Convergent, and that

$$\int_c^\infty \phi(x)\,dx \leq \int_c^\infty f(x)\,dx,$$

and if $|\phi(x)| \geq f(x)$ $\forall\, x$ greater than some number $x_0 > c$ and $\int_c^\infty f(x)\,dx$ is divergent, then $\int_c^\infty \phi(x)\,dx$ is also divergent.



If $f(x)$ Converges, then $\phi(x)$ must also Converge.
If $\phi(x)$ diverges, then $f(x)$ must also diverge.

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(B) Convergence of $\int_c^\infty \frac{dx}{x^n}$, where c is greater than zero.

$$\int_c^\infty \frac{dx}{x^n} = \lim_{x\to\infty} \int_c^x \frac{dx}{x^n} = \lim_{x\to\infty} \left[\frac{x^{1-n}}{1-n}\right]_c^x$$

$$= \lim_{x\to\infty} \frac{1}{1-n}\left[x^{1-n} - c^{1-n}\right]$$

Now following three Cases arises.

Case 1 when $n>1$ then $1-n$ is negative

$$\therefore \lim_{x\to\infty} x^{1-n} = 0$$

$$\therefore \int_c^\infty \frac{dx}{x^n} = \frac{c^{1-n}}{n-1}$$

Hence the integral Converges when $n>1$

Case II If $n<1$, then $1-n$ is +ve

$$\therefore \lim_{x\to\infty} x^{1-n} = \infty$$

$$\therefore \int_c^\infty \frac{dx}{x^n} = \infty$$

Hence the integral diverges when $n<1$.

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Case III If $n = 1$

we have

$$\int_c^{\infty} \frac{dx}{x^n} = \lim_{x\to\infty} \int_c^{x} \frac{dx}{x}$$

$$= \lim_{x\to\infty} \left[\log x - \log c\right]$$

$$= \infty \qquad \left[\because \log \infty = \infty\right.$$

Hence the given integral is divergent when $n = 1$.

Thus the integral $\int_c^x \frac{dx}{x^n}$ Converges when $n > 1$ and diverges when $n \leq 1$.

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Some problems on Comparison Test for the Convergence of Improper Integral.

1) Test the Convergence of the integral

$$\int_0^{\infty} \frac{\cos x}{1+x^2}\, dx .$$

Sol^n Here $\phi(x) = \frac{\cos x}{1+x^2}$

Let $f(x) = \frac{1}{1+x^2}$

Also $\left|\frac{\cos x}{1+x^2}\right| \leq \frac{1}{1+x^2}$ Since

$|\cos x| \leq 1$

Then we have $\int_0^{\infty} \left|\frac{\cos x}{1+x^2}\right| \qquad \int_0^{\infty} \frac{dx}{1+x^2}$

$$= \lim_{x\to\infty} \int_0^{x} \frac{dx}{1+x^2}$$

$$= \lim_{x\to\infty} \left[\tan^{-1} x\right]_0^x$$

$$= \lim_{x\to\infty} \left[\tan^{-1} x - \tan^{-1} 0\right] = \pi/2$$

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$\therefore \int_0^\infty \frac{dx}{1+x^2}$ is Convergent

i.e $\int_0^\infty f(x)\,dx$ is Convergent and hence from Comparison test, it follows that $\int_0^\infty \frac{\cos x}{1+x^2}\,dx$ is also Convergent.

② Show that $\int_a^\infty \frac{dx}{x\sqrt{(1+x^2)}}$ Converges.

<u>Solⁿ</u> Here $\phi(x) = \frac{1}{x\sqrt{1+x^2}}$ and $f(x) = \frac{1}{x^2}$

$\therefore |\phi(x)| < f(x)$

But $\int_a^\infty f(x)\,dx = \int_a^\infty \frac{dx}{x^2}$ is Convergent for $n>1$ & $a>0$ $\left[\int_a^\infty \frac{1}{x^n}dx \text{ is Cgt when } n>1\right]$

Hence by Comparison test $\int_a^\infty \frac{dx}{x\sqrt{x^2+1}}\,dx$ is Convergent.